CONCATENATIONS APPLIED TO ANALYTIC HYPOELLIPTICITY OF OPERATORS WITH DOUBLE CHARACTERISTICS

BY

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ABSTRACT. We use the method of concatenations to get a sufficient condition for a class of analytic pseudodifferential operators with double characteristics to be analytic hypoelliptic.

Introduction. The present paper is concerned with analytic hypoellipticity for operators on an N-dimensional real-analytic manifold Ω, of the form

$$(1) \qquad P(x, D) = I_d p(x, D) + Q(x, D),$$

where I_d is the identity $d \times d$ matrix, $p(x, \xi)$ a scalar analytic symbol, homogeneous of degree m in ξ, and $Q(x, D)$ a $d \times d$ matrix of classical analytic pseudodifferential operators of order $m - 1$ ("classical" means that its total formal symbol is a series of homogeneous terms whose homogeneous degrees drop by integers). We assume that its principal symbol, $p(x, \xi)$, is nonnegative everywhere, that it vanishes exactly of order two on its characteristic set Σ, and that Σ is a symplectic real-analytic submanifold of $T^*\Omega \setminus 0$. For such an operator, analytic hypoellipticity was already obtained in F. Treves [13] under the additional hypothesis that

$$(2) \qquad P(x, D) \text{ is hypoelliptic with loss of one derivative.}$$

Recently, several other studies have been made for the analytic hypoellipticity of similar operators. For example, in [7], G. Métivier extended the result of [13] to the operators with multiple characteristics assuming suitable hypoellipticity in case that the characteristic manifold is symplectic. Whereas, in [6], A. Grigis and L. P. Rothschild gave a necessary and sufficient condition for a class of operators with polynomial coefficients to be analytic hypoelliptic (see also [10] and [11] for a different approach to similar problems).

On the other hand, in [3], L. Boutet de Monvel and F. Treves obtained a necessary and sufficient condition for $P(x, D)$ to be hypoelliptic with loss of one derivative by means of the method of concatenations (introduced first by F. Treves in [12]). According to their work, we can associate with $P(x, D)$ a sequence of operators $P^{(\nu)}$, $\nu \geqslant 0$, with the same principal part as $P(x, D)$, which satisfy certain relations connecting them, called concatenations. In terms of concatenations, the condition (2) can be restated by saying that the "lower order part" of every $P^{(\nu)}$ (i.e., the one

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obtained by neglecting its principal part) is elliptic of order $m - 1$ on Σ. Then the main result of this work is the following:

(3) $P(x, D)$ is analytic hypoelliptic in Ω if the "lower order part" of every $P^{(\nu)}$ is elliptic of arbitrary order less than or equal to $m - 1$ at any point of Σ.

In fact, the latter is equivalent to saying that the "lower order part" of every $P^{(\nu)}$ is elliptic of order $m - 1$ at any point of Σ except possibly one of them. We may even assume that the exceptional one is not necessarily elliptic but has a microlocal analytic parametrix in some conic open neighborhood of any point in Σ.

As an immediate consequence of our result, we derive that $\square_b + C(z, t)$ is analytic hypoelliptic when $\square_b$ is the Kohn-Laplacian on the Heisenberg group H_n and $C(z, t)$ is a real-analytic function on H_n which does not vanish anywhere. This extends the result of E. M. Stein [9] that $\square_b + c$ is analytic hypoelliptic for any nonzero complex number c. Although the above example is only for operators on the Heisenberg group, the main theorem (cf. Theorem 3.1) can also be applied off the group as well.

Notations. We use the standard notations in pseudodifferential operator theory. Functions and distributions in this paper have their values in a large array of finite-dimensional vector spaces over $\mathbf{C}$, which we do not specify but will be clear from the context. Homogeneity always refers to the fibre variable in $T^*\Omega$ and "analytic" always means "real-analytic".

1. Complex coordinates and invariants attached to the operator. Throughout the paper, Ω will be an analytic manifold of dimension N. We shall denote by $T^*\Omega$ the cotangent bundle over Ω, by $\dot{T}^*\Omega$ the complement of the zero section in $T^*\Omega$, and by ω the fundamental symplectic two-form on $T^*\Omega$. In local charts, the coordinates in $T^*\Omega$ always be denoted by $(x, \xi) = (x^1, \ldots, x^N, \xi_1, \ldots, \xi_N)$. For any two smooth complex-valued functions f and g on $T^*\Omega$, H_f and $\{f, g\} = H_f(g)$ stand for the Hamilton field of f and the Poisson bracket of f and g, respectively.

We shall be concerned with analytic pseudodifferential operators with double characteristics of the type

(1.1) $P(x, D) = I_d p(x, D) + Q(x, D),$

where the function $p(x, \xi)$ is a scalar analytic symbol, homogeneous of degree m (real) in ξ, $Q(x, D)$ a classical analytic pseudodifferential operator with values in the space of $d \times d$ complex matrices whose order is at most $m - 1$, and I_d is the identity $d \times d$ matrix. For the theory of analytic symbols and corresponding operators, we refer to [1 and 14]. Furthermore, as in [13], we shall make the following hypotheses:

(1.2) $p(x, \xi)$ is always nonnegative on $T^*\Omega$;

(1.3) $p(x, \xi)$ vanishes exactly of order two on its characteristic set Σ;

(1.4) Σ is an analytic submanifold of $\dot{T}^*\Omega$ and moreover is symplectic, that is, the restriction of ω to its tangent space is nondegenerate everywhere.

Condition (1.4) requires that the dimension of Σ be even and hence so be its codimension: we shall set

$$(1.5) \qquad n = \tfrac{1}{2} \operatorname{codim} \Sigma.$$

Let $\rho = (x^0, \xi_0)$ be an arbitrary point of Σ. Since ω is nondegenerate on $T_\rho(\Sigma)$, the tangent space of Σ at ρ, we have

$$(1.6) \qquad T_\rho(T^*\Omega) = T_\rho(\Sigma) \oplus N_\rho(\Sigma),$$

where $N_\rho(\Sigma)$ denotes the orthogonal complement of $T_\rho(\Sigma)$ in $T_\rho(T^*\Omega)$ with respect to ω. By the assumption (1.3), we can intrinsically define a nondegenerate symmetric bilinear form on $N_\rho(\Sigma)$. The following definition is due to J. Sjöstrand [8] (see also [5]).

DEFINITION 1.1. For each ρ in Σ, we define a bilinear form $q_p(\rho)$ on $T_\rho(T^*\Omega)$ by

$$(1.7) \qquad q_p(\rho)(v_1, v_2) = \tfrac{1}{2} X_1(X_2(p))(\rho), \qquad v_1, v_2 \in T_\rho(T^*\Omega),$$

where X_j are some vector fields defined in a neighborhood of ρ with $X_j(\rho) = v_j$, $j = 1, 2$.

We shall denote by $Q_p(\rho)$ the quadratic form induced by $q_p(\rho)$. Moreover, since ω is nondegenerate, we have an endomorphism $A_p(\rho)$ of $T_\rho(T^*\Omega)$, called the *Hamilton map* of $q_p(\rho)$, defined by

$$(1.8) \qquad q_p(\rho)(v_1, v_2) = \omega(v_1, A_p(\rho)(v_2)), \qquad v_1, v_2 \in T_\rho(T^*\Omega).$$

For convenience, from now on, we shall use q_p and A_p instead of $q_p(\rho)$ and $A_p(\rho)$, respectively. By (1.2) and (1.3), q_p is a well-defined symmetric bilinear form on $T_\rho(T^*\Omega)$ and Q_p is positive-definite when it is restricted to $N_\rho(\Sigma)$. In fact, Q_p is nothing but the quadratic form which begins the Taylor expansion of $p(x, \xi)$ at a point of Σ.

Now, let Γ be an open conic neighborhood of ρ in which there are $2n$ analytic real-valued functions $u_1, \ldots, u_{2n}$, with linearly independent differentials, such that $\Sigma \cap \Gamma$ is exactly defined by the equations

$$(1.9) \qquad u_j(x, \xi) = 0, \qquad j = 1, \ldots, 2n.$$

We take u_j to be homogeneous of degree $\tfrac{1}{2}$ in ξ. Then, in Γ, we can write

$$(1.10) \qquad p(x, \xi) = \sum q_{jk} u_j u_k,$$

where q_{jk} are analytic in Γ and homogeneous of degree $m - 1$ in ξ. We may and shall assume that $q_{jk} = q_{kj}$ for every $j, k = 1, \ldots, 2n$. The following proposition is an analytic version of a particular case of results in [2 and 3].

PROPOSITION 1.1. *There exist complex linear combinations* $z_1, \ldots, z_n$ *of* u_j, $j = 1, \ldots, 2n$, *with complex-valued analytic homogeneous coefficients such that*

$$(1.11) \qquad p(x, \xi) = \sum a_{jk} z_j \bar{z}_k,$$

where the matrix (a_{jk}) *is positive-definite hermitian at each point of* $\Sigma \cap \Gamma$ *and its eigenvalues* $\lambda_j, j = 1,\ldots,n$, *which are all strictly positive, are the invariants attached to the operator* $P(x, D)$;

(1.12)
$$\{ z_j, z_k \} = \frac{1}{\sqrt{-1}} \{ z_j, \bar{z}_k \} - \delta_{jk} = 0 \quad in \ \Gamma$$
$$(possibly \ after \ shrinking \ \Gamma), j, k = 1,\ldots,n.$$

To prove the above proposition, we need some results from linear algebra. Let E be a real vector space of dimension $2n$, ω a real symplectic form on E, and Q a positive-definite quadratic form on E. We shall denote by q the polarization of Q. Extend both q and ω to $E_{\mathbf{C}} = E \otimes \mathbf{C}$, the complexification of E, as bilinear forms. Then we have an endomorphism A of $E_{\mathbf{C}}$, the *Hamilton map* of q, defined by

(1.13)
$$q(u, v) = \omega(u, Av), \qquad u, v \in E_{\mathbf{C}}.$$

For the proof of the following lemma, we refer to [2] (see also [3, 5]).

LEMMA 1.1. *With the above notations, we have that*
(1) *there are n positive real numbers* $\lambda_j, j = 1,\ldots,n$, *not necessarily distinct, such that the eigenvalues of* A *are precisely* $\pm \sqrt{-1} \lambda_j, j = 1,\ldots,n$;
(2) *there exist complex coordinates* $z_1,\ldots,z_n$ *in* $E_{\mathbf{C}}$, *unique up to linear combinations, such that*

(1.14)
$$\omega(z_j, z_k) = \frac{1}{\sqrt{-1}} \omega(z_j, \bar{z}_k) - \delta_{jk} = 0, \qquad j, k = 1,\ldots,n;$$

(1.15)
$Q = \sum a_{jk} z_j \bar{z}_k$, (a_{jk}) *a positive-definite hermitian matrix with* λ_j *in* (1) *as its eigenvalues.*

PROOF OF PROPOSITION 1.1. If $z_k = \sum \alpha_{kj} u_j$, then $dz_k = \sum \alpha_{kj} du_j$ and $\{ z_k, z_l \} = \sum \alpha_{ki} \alpha_{lj} \{ u_i, u_j \}$ on $\Sigma \cap \Gamma$ because u_j vanish on $\Sigma \cap \Gamma$. Applying Lemma 1.1 with $E = \mathbf{R}^{2n}$, Q the quadratic form with the matrix (q_{jk}), and ω the symplectic form whose matrix is the inverse of $(\{ u_j, u_k \})$, we get $z_j, j = 1,\ldots,n$, which satisfy (1.11) and (1.12) only in $\Sigma \cap \Gamma$. But if we denote by x_j and y_j the real and imaginary parts of z_j, (1.12) means

$$\{ x_j, x_k \} = \{ y_j, y_k \} = \{ x_j, y_k \} + \tfrac{1}{2}\delta_{jk} = 0 \quad in \ \Sigma \cap \Gamma, j, k = 1,\ldots,n.$$

Hence, by the classical theorem of Darboux for the real symplectic coordinates, we can choose z_j so that they satisfy (1.12) in Γ if it is sufficiently narrow about $\Sigma \cap \Gamma$.

REMARK 1.1. The vector space spanned by dz_j at each point of $\Sigma \cap \Gamma$ is uniquely determined by (1.11) and (1.12).

PROPOSITION 1.2. *In a possibly smaller conic open set* Γ, *we can construct n analytic pseudodifferential operators* Z_j *with principal symbols* z_j *in* Γ *which satisfy the following commutation relations*:

(1.16)
$[Z_j, Z_k]$ *and* $[Z_j, Z_k^*] - I\delta_{jk}$ *are analytic regularizing in* Γ,
$j, k = 1,\ldots,n$, *where I is the identity operator.*

As usual, the proof is by successive approximations and can be found in [3]. In [2], L. Boutet de Monvel has obtained a more precise result than this using Fourier integral operators in the C^∞-category. For the same result in the analytic category, see [13, Chapter 1].

Let Z_j be the operators in Proposition 1.2 and let A_{jk} be the analytic pseudodifferential operators in Γ with principal symbols a_{jk}. Let us set

$$(1.17) \qquad Z_0 = P(x, D) - I_d\left(\sum A_{jk} Z_k^* Z_j\right) \quad \text{in } \Gamma.$$

Then Z_0 is a $d \times d$ matrix with analytic pseudodifferential operators in Γ as entries whose order is at most $m - 1$. We shall denote by $\sigma_0 = \sigma_0(x, \xi)$ the restriction to $\Sigma \cap \Gamma$ of the principal symbol of Z_0 regarded as an operator of order $m - 1$ in Γ. If we set

$$(1.18) \qquad W_j = \sum A_{jk} Z_k^*, \qquad j = 1, \dots, n,$$

then we have, in Γ,

$$(1.19) \qquad P(x, D) = I_d\left(\sum W_j Z_j\right) + Z_0.$$

Let $p_m + p_{m-1} + \cdots$ be the complete symbol of P in a local coordinate system (x, ξ), where $p_j(x, \xi)$ is homogeneous of degree j in ξ. Then

$$p_{m-1} = \sigma_0(x, \xi) + I_d\left(\frac{1}{\sqrt{-1}} \sum \frac{\partial w_j}{\partial \xi_k} \frac{\partial z_j}{\partial x^k}\right)$$

$$= \sigma_0(x, \xi) + \frac{1}{2\sqrt{-1}} \sum \frac{\partial^2 p_m(x, \xi)}{\partial \xi_k \partial x^k} + \frac{1}{2\sqrt{-1}} I_d\left(\sum \{w_j, z_j\}\right)$$

$$= \sigma_0(x, \xi) + \frac{1}{2\sqrt{-1}} \sum \frac{\partial^2 p_m(x, \xi)}{\partial \xi_k \partial x^k} - I_d\left(\frac{1}{2} \sum \lambda_j\right) \quad \text{on } \Sigma \cap \Gamma,$$

where w_j are the principal symbol of W_j and λ_j are the eigenvalues of the matrix (a_{jk}) (cf. Proposition 1.1). Therefore we have another invariant of $P(x, D)$:

$$(1.20) \qquad \sigma_0(x, \xi) = \sigma_{\text{sub}}(P) + I_d\left(\tfrac{1}{2} \sum \lambda_j\right),$$

where

$$\sigma_{\text{sub}}(P) = p_{m-1}(x, \xi) - \frac{1}{2\sqrt{-1}} \sum \frac{\partial^2 p_m(x, \xi)}{\partial \xi_k \partial x^k}$$

is the subprincipal symbol of $P(x, D)$. In fact, this invariant is a function defined on Σ, whose value lies in the space of $d \times d$ complex matrices.

REMARK 1.2. Note that the adjoint P^* of P is an operator of exactly the same type as P and the invariants associated with P^* are the complex conjugates of those associated with P.

2. Concatenations and hypoellipticity with loss of one derivative. Throughout this section, we restrict our attention to the conic open set Γ in which we have (1.19). We are going to construct a sequence of operators, which satisfy certain relations, called *concatenations*, through which (analytic) hypoellipticity can be transmitted backward

and then give the connection between the concatenations and the hypoellipticity with loss of one derivative. The latter means, in our case, that for any open subset U of Ω, any real number s, and any distribution u in U,

$$(2.1) \qquad P(x, D)(u) \in H^s_{\text{loc}}(U; \mathbf{C}^d) \quad \text{implies} \quad u \in H^{s+m-1}_{\text{loc}}(U; \mathbf{C}^d).$$

We mainly follow the arguments in [3] (see also [12 and 13] for the method of concatenations). We shall now make the additional hypothesis that

$$(2.2) \qquad Z_0 \text{ is elliptic (of arbitrary order} \leqslant m - 1) \text{ in } \Gamma.$$

Let Z_0^{-1} be the inverse of Z_0 in Γ (in the algebra of analytic pseudodifferential operators) and let us form

$$(2.3) \qquad Z_j' = Z_0 Z_j Z_0^{-1}, \qquad j = 1, \ldots, n.$$

Then we have

$$
\begin{aligned}
Z_j'P &= \left(\sum Z_0 Z_j Z_0^{-1} W_k Z_k \right) + Z_0 Z_j \\
&= I_d \left(\sum Z_j W_k Z_k \right) + Z_0 Z_j + \sum \left(Z_0 [Z_j, Z_0^{-1}] W_k \right) Z_k \\
&= \left\{ I_d \left(\sum W_k Z_k \right) + Z_0 \right\} Z_j + \sum \left(I_d [Z_j, W_k] + Z_0 [Z_j, Z_0^{-1}] W_k \right) Z_k,
\end{aligned}
$$

where " = " must be understood in the sense of being equal modulo analytic regularizing operators in Γ. If we set $P_{jk} = \delta_{jk} P + I_d[Z_j, W_k] + Z_0[Z_j, Z_0^{-1}]W_k$, then we have

$$(2.4) \qquad Z_j'P = \sum P_{jk} Z_k, \qquad j = 1, \ldots, n.$$

Let $P^{(1)}$ denote the $nd \times nd$ system (P_{jk}) and Z (resp. $Z^{(1)}$) the column with n entries, which are $d \times d$ systems $I_d Z_j$ (resp. $I_d Z_j'$). Then we can rewrite (2.4) in the form

$$(2.5) \qquad Z^{(1)}P = P^{(1)}Z.$$

On the other hand, we can write

$$(2.6) \qquad P^{(1)} = I_{nd} \left(\sum W_k Z_k \right) + Z_0^{(1)},$$

where $Z_0^{(1)}$ is of order at most $m - 1$ in Γ.

Now, assuming that $Z_0^{(1)}$ is also elliptic in Γ, we can repeat the same argument as above inductively and get a sequence of operators $P^{(\nu)}$, $\nu \geqslant 0$, with $P^{(0)} = P$, which form concatenations

$$(2.7) \qquad Z^{(\nu+1)}P^{(\nu)} = P^{(\nu+1)}Z,$$

where

$$(2.8) \qquad \begin{aligned} &Z \text{ is the column with } n \text{ entries, respectively equal to } I_n \nu_d Z_j, \\ &j = 1, \ldots, n, \end{aligned}$$

$$(2.9) \qquad Z_j^{(\nu+1)} = Z_0^{(\nu)} Z_j \left(Z_0^{(\nu)} \right)^{-1}, \qquad j = 1, \ldots, n,$$

$$(2.10) \qquad P^{(\nu)} = I_n \nu_d \left(\sum W_k Z_k \right) + Z_0^{(\nu)} \qquad \left(Z_0^{(0)} = Z_0 \right).$$

Of course, in the above arguments, we have assumed that

$$(2.11) \qquad Z_0^{(\mu)} \text{ is elliptic (of arbitrary order} \leqslant m - 1) \text{ in } \Gamma, \mu = 0, \ldots, \nu.$$

Our argument might require successive shrinkings of Γ, which does not matter to us since we shall use only a finite number of steps in the concatenations. We are now ready to give the connection between the concatenations and the hypoellipticity with loss of one derivative for $P(x, D)$. Let us consider a special case of (2.11):

(2.12) $Z_0^{(\mu)}$ is elliptic of order $m - 1$ in Γ.

In this case, (2.12) is equivalent (possibily after shrinking Γ) to

(2.13) $Z_0^{(\mu)}$ is elliptic of order $m - 1$ in $\Sigma \cap \Gamma$.

Let us denote by $\sigma_0^{(\mu)}$ the restriction to $\Sigma \cap \Gamma$ of the principal symbol of $Z_0^{(\mu)}$ as an operator of order $m - 1$ and set

(2.14) $\theta_{jk} = \sigma\left(\left[Z_j, W_k\right]\right)|\Sigma \cap \Gamma, \qquad j, k = 1, \ldots, n.$

For $\mu = 1$, from the definition of P_{jk}, we have

(2.15) $\left(\sigma_0^{(1)}\right)_{jk} = \delta_{jk}\sigma_0^{(0)} + \theta_{jk}I_d, \qquad j, k = 1, \ldots, n.$

That is,

(2.16) $\sigma_0^{(1)} = \sigma_0^{(0)} \otimes I_n + I_d \otimes \Theta,$

where Θ is the matrix (θ_{jk}). More generally, for any $\mu \geqslant 1$,

(2.17) $\sigma_0^{(\mu)} = \sigma_0^{(\mu-1)} \otimes I_n + I_{n^{\mu-1}d} \otimes \Theta.$

The hypothesis (2.13) simply means that the eigenvalues of $\sigma_0^{(\mu)}$, which depends on the points of $\Sigma \cap \Gamma$, are all distinct from zero. From (2.17), we can easily see that any eigenvalue of $\sigma_0^{(\mu)}$ is a sum of an eigenvalue of $\sigma_0^{(\mu-1)}$ and an eigenvalue of Θ. By induction on μ, we have

(2.18) $\mathrm{Spec}\ \sigma_0^{(\mu)} = \mathrm{Spec}\ \sigma_0^{(0)} + \underbrace{\mathrm{Spec}\ \Theta + \cdots + \mathrm{Spec}\ \Theta}_{\mu\text{-times}},$

where the plus signs on the right stand for the addition of complex numbers. On the other hand, from (1.18) and (2.14), we see that the matrix Θ is just the transpose of (a_{jk}) over $\Sigma \cap \Gamma$ and hence the eigenvalues of Θ are the positive real numbers λ_j, $j = 1, \ldots, n$, introduced in Proposition 1.1.

Therefore, when μ ranges over Z_+ and Γ ranges over a covering of Σ by open cones, the hypothesis (2.13) is equivalent to the following property:

(2.19) For no point (x, ξ) in Σ and for no n-tuple of nonnegative integers $r = (r_1, \ldots, r_n)$, does the $d \times d$ matrix $\sigma_0(x, \xi)$ (let us remind that $\sigma_0(x, \xi)$ is an invariant of $P(x, D)$ defined on Σ) have the eigenvalues $-\Sigma r_j \lambda_j$.

As is now well known (cf. [2, 3, 5, 8]), condition (2.19) is necessary and sufficient for $P(x, D)$ to be hypoelliptic with loss of one derivative.

REMARK 2.1. The above construction of the concatenations and the observations for the condition (2.13) have already been done in [3] for a larger class of operators in the C^∞-category. However, as long as we are concerned with analytic hypoellipticity, we can also construct the same concatenations (2.7) under a much weaker condition that every $Z_0^{(\mu)}$ has an analytic parametrix in some open conic neighborhood of each point of Σ.

REMARK 2.2. According to Remark 1.2, P is hypoellitpic with loss of one derivative if and only if so is the adjoint P^* of P.

3. Analytic hypoellipticity. Throughout this section, we are going to assume that $P(x, D)$ is given by (1.19) not only in Γ but in the whole of $T^*\Omega$. This is possible since analytic hypoellipticity is purely a local matter and outside Σ, $P(x, D)$ is elliptic. Moreover, after a preliminary shrinking of Γ if necessary, we may and shall assume that Σ is conically compact. Now, let us consider the following condition for $P(x, D)$ under which we can construct concatenations (2.7):

(3.1)   Every $Z_0^{(\nu)}$, $\nu \geqslant 0$, has a microlocal analytic parametrix in some conic open neighborhood of each point in Σ.

It follows from the formula (2.18) and the conical compactness of Σ that all but a finite number of $Z_0^{(\nu)}$ are elliptic of order $m - 1$ on Σ when $P(x, D)$ satisfies condition (3.1). Now, we can state and prove the main result of this work.

THEOREM 3.1. *If $P(x, D)$ satisfies condition* (3.1), *then $P(x, D)$ is analytic hypoelliptic in* Ω.

First of all, let us recall the main result of [**13**].

(3.2)   Let $P(x, D)$ be given by (1.1) which satisfies (1.2), (1.3), and (1.4). If it is hypoelliptic with loss of one derivative, then any point of Σ has an open conic neighborhood, say Γ^0, in which $P(x, D)$ is analytic hypoelliptic, that is, for any $u \in \varepsilon'(\Omega : \mathbf{C}^d)$, $WF_a(u) \cap \Gamma^0 = WF_a(Pu) \cap \Gamma^0$ (cf. [**13**, Theorem 1.4.2]), where $WF_a(u)$ stands for the *analytic wave-front set* of u.

(The proof is by construction of a microlocal analytic parametrix of $P(x, D)$ near every point of Σ.)

Then, recalling that the base projection of analytic wave-front set of a distribution is its analytic singular-support, the analytic hypoellipticity of $P(x, D)$ is just a local version of the above result.

The following lemma will show that the analytic hypoellipticity can be transmitted backward along the concatenations.

LEMMA 3.1. *Suppose $P(x, D)$ satisfies condition* (3.1). *If, for some* $\mu \geqslant 0$, $P^{(\mu+1)}$ *is analytic hypoelliptic in* Ω, *then so is* $P^{(\mu)}$.

PROOF. Let u be a distribution with compact support in Ω for which $P^{(\mu)}(u)$ is analytic in an open set U in Ω. Since the characteristic set of $P^{(\mu)}$ is exactly Σ, $WF_a(u) \cap T^*U$ is contained in Σ. On the other hand, from (2.7), $P^{(\mu+1)}Z(u) = Z^{(\mu+1)}P^{(\mu)}(u)$ and so $Z(u)$ is analytic in U because $P^{(\mu+1)}$ is analytic hypoelliptic in Ω. That means, from the definition of Z (cf. (2.8)), all the $Z_j(u)$ are analytic in U. Therefore,

$$Z_0^{(\mu)}(u) = P^{(\mu)}(u) - I_n \mu_d \left(\sum W_k Z_k \right)(u)$$

(cf. (2.10)) is also analytic in U. But since $Z_0^{(\mu)}$ is analytic hypoelliptic near every point of Σ by the hypothesis (3.1), we have $WF_a(u) \cap T^*U \cap \Sigma = \varnothing$, which implies

that $WF_a(u) \cap T^*U = \varnothing$ (recalling that $WF_a(u) \cap T^*U$ is contained in Σ). That is, u is also analytic in U.

REMARK 3.1. Repeated applications of Lemma 3.1 show that if $P(x, D)$ satisfies condition (3.1) and if $P^{(\mu+1)}$ is analytic hypoelliptic in Ω for some $\mu \geqslant 0$, then so are $P = P^{(0)}, \ldots, P^{(\mu)}$.

PROOF OF THEOREM 3.1. As we have mentioned in the beginning of this section, under the condition (3.1), there is a nonnegative integer μ_0 such that all the $Z_0^{(\mu)}$, $\mu > \mu_0$, are elliptic of order $m - 1$ on Σ. Then, in particular, $P^{(\mu_0+1)}$ satisfies condition (2.13) and hence is hypoelliptic with loss of one derivative since it has the same scalar principal part as that of $P(x, D)$. Therefore, thanks to (3.2), $P^{(\mu_0+1)}$ is analytic hypoelliptic in Ω. Due to Remark 3.1, this proves that $P(x, D)$ itself is analytic hypoelliptic in Ω.

Noting that condition (2.11) is a special case of (3.1), we get immediately the following corollary.

COROLLARY 3.1. *If $P(x, D)$ satisfies condition (2.11), then $P(x, D)$ is analytic hypoelliptic in Ω. Moreover the condition (2.11) is satisfied if Z_0 is elliptic of order strictly less than $m - 1$ in some conic open neighborhood of each point in Σ.*

PROOF. We only need to prove the second statement of the above corollary. Let us consider $\sigma_0^{(\mu)}$, $\mu \geqslant 0$, which is the restriction to Σ of the principal symbol of $Z_0^{(\mu)}$ as an operator of order $m - 1$. Then $\sigma_0^{(0)}$ is identically zero on Σ by the assumption. Therefore, from (2.18), any eigenvalue of $\sigma_0^{(\mu)}$, $\mu > 0$, is of the form $\sum r_j \lambda_j$, where $r = (r_1, \ldots, r_n) \in Z_+^n$, $|r| = r_1 + \cdots + r_n = \mu$, and λ_j are the eigenvalues of (a_{jk}) (cf. (1.11)). Since all the λ_j are strictly positive, 0 cannot be an eigenvalue of $\sigma_0^{(\mu)}$ for $\mu > 0$. That is, all the $Z_0^{(\mu)}$, $\mu > 0$, are elliptic of order $m - 1$ on Σ. Hence condition (2.11) is satisfied (cf. (2.13)).

REMARK 3.2. Although our main concern is analytic hypoellipticity, we incidently get C^∞-hypoellipticity of $P(x, D)$, without any significant change of previous arguments, under the same hypotheses as in Theorem 3.1 or Corollary 3.1 except the fact that $P(x, D)$ can be a usual (C^∞-) pseudodifferential operator instead of an analytic one.

4. Application. In this section, as an application of our results, we will consider an operator defined on the Heisenberg group H_n which arises as the boundary of the generalized upper half-space in $\mathbb{C}^{n+1}$,

$$D = \left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}; \ \mathrm{Im} \ z_{n+1} > |z_1|^2 + \cdots + |z_n|^2 \right\}.$$

Its boundary can be identified with $\mathbb{C}^n \times \mathbb{R}^1$, which has a structure of Heisenberg group H_n on which we have the following left-invariant complex vector fields (denoting by $(z, t) = (z_1, \ldots, z_n, t)$ the coordinates in H_n):

$$(4.1) \qquad Z_j = \frac{\partial}{\partial z_j} + \sqrt{-1} \, \bar{z}_j \frac{\partial}{\partial t},$$

$$\bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \sqrt{-1} \, z_j \frac{\partial}{\partial t}, \qquad j = 1, \ldots, n,$$

which satisfies the following commutation relations:

(4.2) $[Z_j, Z_k] = 0,$ $[Z_j, Z_k^*] = -2D_t\delta_{jk},$ $j, k = 1, \ldots, n.$

Under the identification of H_n with ∂D, $\overline{Z}_j$ annihilate holomorphic functions restricted to ∂D and hence induce a complex $\overline{\partial}_b$, called the *tangential Cauchy-Riemann complex* on H_n. It is defined on the functions on H_n by the formula

(4.3) $\overline{\partial}_b f = \sum (\overline{Z}_j f) d\overline{z}_j,$

which extends as a derivation mapping $(0, q)$-forms to $(0, q + 1)$-forms on H_n. Then we have the (Kohn-) Laplacian $\square_b$ associated to the $\overline{\partial}_b$-complex

(4.4) $\square_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b,$

where $\overline{\partial}_b^*$ is the formal adjoint of $\overline{\partial}_b$ with respect to L^2-inner product. As is well known, the restriction of $\square_b$ to the functions on H_n, which is given by

(4.5) $\square_b = -\sum Z_j \overline{Z}_k,$

is neither locally solvable nor hypoelliptic (see, for example, [4]) and its characteristic manifold Σ is symplectic, which is a reflection of the fact that H_n is strongly pseudoconvex.

In [9], E. M. Stein restricted his attention to the left-invariant operators on H_n and showed that $\square_b + c$ is C^∞- and analytic hypoelliptic for any nonzero complex number c when it acts on the functions on H_n. We see that one gets the following as an immediate consequence of Corollary 3.1 and Remark 3.2.

THEOREM 4.1. *Let $C(z, t)$ be any real-analytic (resp. C^∞-) complex-valued function defined on H_n. If it is nowhere vanishing on H_n, then the operator $\square_b + C(z, t)$ is analytic hypoelliptic (resp. C^∞-hypoelliptic) on H_n.*

PROOF. Let us denote by P the operator $\square_b + C(z, t)$. In order to apply Corollary 3.1, we need the commutation relations (1.16) rather than (4.2). Since

$$D_t = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial t}$$

is elliptic on Σ, in a conic open neighborhood of any point in Σ, we can replace Z_j (cf. (4.1)) by $(2|D_t|)^{-1/2} Z_j$ so that they satisfy (1.16) (up to sign). It suffices to prove the theorem for $(2|D_t|)^{-1} P$, which follows immediately from Corollary 3.1.

REMARK 4.1. We can also consider analytic hypoellipticity of $\square_b$ acting on $(0, q)$-forms on H_n for $q > 0$. As is well known, $\square_b = IL_\alpha$, where $\alpha = n - 2q$ when it acts on $(0, q)$-forms,

$$L_\alpha = -\frac{1}{2}\sum (Z_j \overline{Z}_j + \overline{Z}_j Z_j) + \sqrt{-1}\,\alpha T \qquad \left(T = \frac{\partial}{\partial t}\right),$$

and I is the identity matrix of an appropriate degree which depends on q. Moreover, according to (2.19), L_α is hypoelliptic with loss of one derivative if and only if $n \pm \alpha$ is not an even integer $\leqslant 0$. Hence $\square_b$ is analytic hypoelliptic when it acts on $(0, q)$-forms for $q \neq 0, n$. This is already pointed out by F. Treves in [13] (see also [10 and 11] for a different proof (using L^2-methods) of analytic hypoellipticity of $\square_b$).

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